ON THE SPECTRUMS OF ERGODIC SCHRODINGER OPERATORS WITH FINITELY VALUED POTENTIALS

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ABSTRACT. We show that the Lebesgue measure of the spectrum of ergodic Schrödinger operators with potentials defined by non-constant function over any minimal aperiodic finite subshift tends to zero as the coupling constant tends to infinity. We also obtained a quantitative upper bound for the measure of the spectrum. This follows from a result we proved for ergodic Schrödinger operators with potentials generated by aperiodic subshift under two conditions on the recurrence property of the subshift. We also show that one of these conditions is necessary for such result.

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1. INTRODUCTION

This paper is motivated by Simon's subshift conjecture (in [10], see also [5]) and the desire to get a better understanding of recently discovered counter-examples in [1]. Consider an aperiodic strictly ergodic subshift over a finite alphabet, which is assumed to consist of real numbers for simplicity, consider the Schrödinger operators in $\ell^2(\mathbb{Z})$ with potentials given by the elements of the subshift. By minimality, the spectrum is the same for every element in the subshift.

The common spectrum was suspected to be of zero Lebesgue measure. For CMV matrices, Barry Simon conjectured the following in [10].

CONJECTURE 1. *Given a minimal subshift of Verblunsky coefficients which is not periodic, the common essential support of the associated measures has zero Lebesgue measure.*

There is also a Schrödinger version of the subshift conjecture (see [1]),

CONJECTURE 2. *Given $\mathcal{A} \subset \mathbb{R}$ finite and a minimal subshift $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ which is not periodic, the associated common spectrum has zero Lebesgue measure.*

It has been shown that for strictly ergodic subshifts satisfying the so-called Boshernitzan condition, the Schrödinger operators have zero-measure spectrum for any non-constant potentials [6], and for CMV matrices, one has zero-measure supports [7]. More results on subshifts associated operators can be found in [5].

In the recent work of Avila, Damanik and Zhang [1], the subshift conjecture is shown to be false, for both Schrödinger version and the orginal version for CMV matrices. In fact, the authors proved the following theorem for Schrödinger operators (Theorem 1 in [1])

Theorem 1. *Given $\mathcal{A} \subset \mathbb{R}$ with $2 \leq card\mathcal{A} < \infty$, there is a minimal subshift $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ which is not periodic, such that the associated spectrum $\Sigma \subset \mathbb{R}$ has strictly positive Lebesgue measure.*

They also proved a CMV matrices analog (Theorem 2 in [1]) which disproved the subshift conjecture in its original formulation.

In [1], the authors also proved a positive result roughly saying that when the system endowed with an ergodic invariant measure is relatively simple, the associated density of states measure is purely singular. The precise condition is formulated as being "almost surely polynomially transitive" and "almost surely of polynomial complexity". This theorem works for subshifts generated by translations on tori with Diophantine frequencies, certain skew shifts and interval exchange transformations. Note that this theorem does not imply that the measure of the spectrum is zero.

Given this new phenomenon, namely that subshift generated potentials can give positive-measure spectrum, the following question arises naturally.

Question 1. *Given a minimal aperiodic subshift and a non-constant potential function, how large can the Lebesgue measure of the spectrum be ?*

This paper is an attempt to study this question. The main result is the following.

Theorem 2. *Given any $k \geq 2$, a minimal aperiodic subshift $\Omega \subset \{1, \cdots, k\}^{\mathbb{Z}}$. Then for any $0 < \gamma < \frac{1}{4}$ the following is true. For any non-constant function $v : \{1, \cdots, k\} \to \mathbb{R}$, there exists $C > 0$, such that for any $\lambda > 0$, the Lebesgue measure of the spectrum of the Schrödinger operator with potential λv is smaller than $C\lambda^{-\gamma}$.*

We actually proved the following more general result for ergodic Schrödinger operators with shift-generated potentials

Theorem 3. *Given any $k \geq 2$, an aperiodic subshift $\Omega \subset \{1, \cdots, k\}^{\mathbb{Z}}$ endowed with an ergodic shift invariant measure μ, such that : (1) there exists an integer $K > 0$ such that $\mu(\{\omega; \omega_0 = \omega_1 = \cdots = \omega_{K-1}\}) = 0$; (2) there exists an integer $L > 0$ such that for any $1 \leq i \leq k$, any $\omega = (\omega_p)_{p \in \mathbb{Z}} \in supp\mu$, there exists $0 \leq j \leq L - 1$ such that $\omega_j = i$. Then for any $0 < \gamma < \frac{1}{4}$, there exists a constant $C > 0$, such that for any non-constant function $v : \{1, \cdots, k\} \to \mathbb{R}$, denote $\lambda = \min(|v(i) - v(j)|; 1 \leq i < j \leq k)$, then $Leb(\Sigma_v) < C\lambda^{-\gamma}$. Here Σ_v denotes the almost sure spectrum with potential v.*

In fact, we will prove a better bound for the exponent γ based on more detailed knowledge of the recurrence property of the subshift.

Since any minimal subshift Ω, any ergodic shift invariant measure μ on Ω satisfy condition (1),(2) in Theorem 3, Theorem 2 follows as an immediate corollary.

To the best of the author's knowledge, this result seems to be the first non-trivial upper bound for the Lebesgue measure of the spectrum for this class of Schrödinger operators without any complexity bound assumption.

We note that if one only assumes the conditions of Theorem 3, one cannot hope to prove zero-measure spectrum for all sufficiently sparse potentials. In fact we have the following theorem which is a slight modification of Theorem 1 in [1].

Theorem 4. *Given any $k \geq 2$, $\epsilon > 0$, any countable subset B of non-constant functions from $\mathcal{A}$ to $\mathbb{R}$. There exists $C > 0$, a minimal aperiodic subshift $\Omega \subset \{1, \cdots, k\}^{\mathbb{Z}}$ with complexity function p satisfying $p(n) < Cn^{1+\epsilon}, \forall n \in \mathbb{N}$, such that for any $v \in B$, the Schrödinger operator with potential v has spectrum of strictly positive Lebesgue measure.*

Here for any $n \geq 1$, the complexity function $p(n)$ denote the number of different words of length n appeared in the subshift. This notion can also be found in many literatures on Schrödinger operators with shift-generated potentials, for example [1], [5] and [6].

We also note that the condition (1) in Theorem 3 is necessary to ensure that the measure of the spectrum tends to zero as the "sparseness" of the potential function grows to infinity. This is seen from the following theorem, which seems to be folklore.

Theorem 5. *Given any $k \geq 2$, a subshift $\Omega \subset \{1, \cdots, k\}^{\mathbb{Z}}$, an ergodic shift invariant measure μ such that there exists $i \in \{1, \cdots, k\}$ such that for any integer $N > 0$, $\mu(\{\omega; \omega_0 = \omega_1 = \cdots = \omega_{N-1} = i\}) > 0$. Then for any function $v : \{1, \cdots, k\} \to \mathbb{R}$, we have $[-2 + v(i), 2 + v(i)] \subset \Sigma_v$. Here Σ_v denotes the almost sure spectrum with potential v.*

1.1. **Outline of the proof.** As mentioned above, the subshift conjecture is true for many subshifts. As discussed in [5], two principal approachs for establishing zero-measure spectrum are: 1. Using trace map dynamics; 2.Proving uniform convergence, usually under Boshernitzan's condition. In both cases, one first show that the spectrum coincides with the set of energy on which the Lyapunov exponent vanish, then apply Kotani's theory [9]. Thus in these approaches, one comes down to showing that non-uniformly hyperbolicity does not appear at all.

In order to prove our result, we have to consider the possible appearance of non-uniformly hyperbolic dynamics.Then the main task is to show that the set of energy corresponding to non-uniformly hyperbolic dynamics has small measure. Instead of directly establishing uniformly hyperbolicity for many energies, we appeal to Berezansky's theorem in the spectral theory of lattice Schrödinger operators, which says that for almost every energy with respect to the spectral measure, there exists a generalised eigenfunction with polynomial growth. We will construct a closed subset $J \subset \mathbb{R}$ of small Lebesgue measure and a subset Ω' of the shift space of positive measure, such that for element $\omega \in \Omega'$, for any energy outside of this closed set, the Schrödinger operator associated to ω has no generalised eigenfunction of polynomial growth. This approach concerning the generalised eigenfunction is inspired by the proof of Theorem 3 in [1].

The main technical difficulty with this naive approach is that : We still have to consider dynamics associated with different energy, whose longtime behaviours could be very different. We overcome this difficulty using the so-called Benedicks-Carleson argument that is originated in the study of Hénon maps. It was introduced to

the study of quasi-periodic cocycles by Young [12], who showed among other things that for certain parametrised family of quasi-periodic cocycles, the Lyapunov exponents are large for a large set of parameters. More recent developments of this type of arguments can by found in [2],[11]. Our main observation is that Benedicks-Carleson arguments provide a unified mechanism for hyperbolicity for all the energy that is not removed from the parameter exclusion. Roughly speaking, for a short interval of energy that could cause non-uniformly hyperbolicity, we have only one "bad" alphabet that could ruin the exponential growth of the associated cocycle. We inductively define a nested sequence of subset of the subshift starting this alphabet, so that $(n+1)-$th set is contained in $n-$th set, and each time we consider the Poincaré return map restricted to $(n+1)-$set and form an accelerated cocycle defined over $(n+1)-$th set, which is just the consecutive multiplication along the first return map. We inductively prove that the accelerated cocycles are highly hyperbolic and the most expanding and most contracting directions can be related to those of the previous accelerated cocycles. The only problem occurs when apply the matrix corresponding to the "bad" alphabet. We then remove a set of energy each time to produce certain amount of transversality. For the remains of energies, the corresponding Schrödinger cocycles are exponentially increasing along a subsequence in time (this can be compared to one of the main results in [13], which says that a cocycle is uniformly exponentially increasing is equivalently to being uniformly hyperbolic). Since we can get good control of the closeness of the stable/unstable directions for matrices in consecutive steps, the parameter removed in each step stays close to the parameters removed in the previous step. Finally, we find a subset of the subshift with positive measure whose elements have good forward and backward landing time at arbitrarily large time scale, which would preclude the existence of generalised eigenfunctions of polynomial growth.

1.2. **Structure of the paper.** In Section 2, we introduce the setting and the notations. We also show an *a priori* bound for the spectrum based on a classical theorem of Johnson. In Section 3, we introduce a sequence of objets and parameters that will later help us estimate the spectrum and control the dynamics. In Section 4 we deal with a technical lemma that will be used repeatedly in Section 5. Section 5 is devoted to the construction and estimation of the objets introduced in Section 3. In Section 6, we relate the objets introduced in Section 3 to the spectrum, which is the main novelty of this paper.

In Section 7, we estimate the spectrum and conclude the proof of the main theorem. In Section 8, we prove Theorem 4 and Theorem 5.

2. Ergodic Schrödinger operators over subshifts

Given a finite set $\mathcal{A}$, we define the shift transformation T on $\mathcal{A}^{\mathbb{Z}}$ by $T(\omega)_n = \omega_{n+1}$. Let Ω be a $T-$invariant compact subset of $\mathcal{A}^{\mathbb{Z}}$. Let $\mu \in \mathcal{P}(\Omega)$ be an ergodic $T-$invariant measure. Without loss of generality, in this paper we will always assume that

$$\Omega = supp\mu$$

for otherwise we can replace Ω by $supp\mu$. We will assume that for any $\alpha \in \mathcal{A}$, we have

$$\mu(\{\omega; \omega_0 = \alpha\}) > 0$$

for otherwise we can replace $\mathcal{A}$ by one of its subsets.

Let $v : \mathcal{A} \to \mathbb{R}$ be a function. Without loss of generality, in this paper we will always assume that: for any $\alpha, \beta \in \mathcal{A}$, we have $v(\alpha) \neq v(\beta)$. To each such v, we can associate a continuous function $V : \Omega \to \mathbb{R}$ defined by $V(\omega) = v(\omega_0)$. In the study of ergodic Schrödinger operators, V is usually referred to as the potential function. In the following, we will call both V and v the potential without causing ambiguity in understand the results. For each $\omega \in \Omega$, let Σ_ω denote the spectrum of the Schrödinger operator H_ω on $\ell^2(\mathbb{Z})$ defined by

$$(2.1) \qquad (H_\omega u)_n = u_{n+1} + u_{n-1} + V(T^n\omega)u_n$$

It is well-known that Σ_ω is the same for μ almost every ω. We denote the almost sure spectrum of this family of operators by Σ_v. When there is no confusion on the potential function v, we denote $\Sigma = \Sigma_v$. It is also well-known that when (Ω, T) is minimal, Σ_ω is the same for all ω. Although we will not exploit this fact in this paper.

Denote $R = \{v(\alpha)\}_{\alpha \in \mathcal{A}}$. For any $\alpha \in \mathcal{A}$, we denote

$$A_\alpha^E = \begin{bmatrix} E - v(\alpha) & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$A^E(\omega) = A_{\omega_0}^E$$

We define a function $A^E : \mathbb{Z} \times \Omega \to SL(2, \mathbb{R})$ by setting

$$A^E(0, \omega) = Id$$
$$A^E(k, \omega) = A^E(T^{k-1}(\omega)) \cdots A^E(\omega) \text{ for all } k > 0$$

and

$$A^E(-k, \omega) = A^E(T^{-k}(\omega))^{-1} \cdots A^E(T^{-1}(\omega))^{-1} \text{ for all } k > 0$$

For any $n, m \geq 0$, any $\omega \in \Omega$ we have the following relation

$$A^E(n + m, \omega) = A^E(m, T^n(\omega))A^E(n, \omega)$$

For any finite word $\alpha = \omega_0 \omega_1 \cdots \omega_{n-1}$, where $\omega_i \in \mathcal{A}$ for all $0 \leq i \leq n - 1$, we define

$$A^E(\alpha) = A_{\omega_{n-1}}^E \cdots A_{\omega_0}^E$$

Definition 1. *For any function $v : \mathcal{A} \to \mathbb{R}$, we call v an admissible potential if for any two distinct elements $\alpha, \beta \in \mathcal{A}$, we have $v(\alpha) \neq v(\beta)$. For any admissible potential v, we denote $\lambda_v = \min_{\alpha, \beta \in \mathcal{A}, \alpha \neq \beta} |v(\alpha) - v(\beta)|$, and call it the* sparseness constant *of the potential v.*

We have the following notion called "Uniformly Hyperbolic". We use the definition in [13], adapted to our situation.

Definition 2. *Fix an admissible potential $v : \mathcal{A} \to \mathbb{R}$, for each $E \in \mathbb{R}$, we have a map $A^E(1, \cdot) : \Omega \to SL(2, \mathbb{R})$, and we call it the* Schrödinger cocycle *at energy E. The Schrödinger cocycle at energy E is called* Uniformly Hyperbolic *if there exists two (necessarily unique) invariant continuous sections*

$$e_s, e_u : \Omega \to \mathbb{P}^1 \mathbb{R}^2$$

with $e_s(\omega) \neq e_u(\omega)$ for any $\omega \in \Omega$, and e_s is uniformly repelling (in the $\mathbb{P}^1 \mathbb{R}^2$ direction) and e_u is uniformly contracting (in the $\mathbb{P}^1 \mathbb{R}^2$ direction).

We have the following well-known result (see [8])

Theorem 6 (Johnson). *We have $\Sigma = \{E; \ A^E(1, \cdot) \text{ is not Uniformly Hyperbolic}\}$*

For any $E_0 \in R$, we associate an interval centered at E_0

$$I_{E_0} = [E_0 - H, E_0 + H]$$

for some constant $H > 0$ to be determined as follow.

We choose $H > 0$ such that, for any $E \notin \bigcup_{E_0 \in R} I_{E_0}$, $A^E(1, \cdot)$ is Uniformly Hyperbolic. Indeed, when H is sufficiently large, for any $E \notin \bigcup_{E_0 \in R} I_{E_0}$, there exist two closed cones $C_+, C_- \subset \mathbb{R}^2$ such that for any $\alpha \in \mathcal{A}$, we have $A_\alpha^E(C_+) \setminus \{0\} \subset intC_+$ and $(A_\alpha^E)^{-1}(C_-) \setminus \{0\} \subset intC_-$. A classical construction in dynamical systems shows that this implies $A^E(1, \cdot)$ is Uniformly Hyperbolic.

Hence by Theorem 6

$$(2.2) \qquad \Sigma \subset \bigcup_{E_0 \in R} I_{E_0}$$

We will need the following general result on lattice Schrödginer operators.(see [4])

Theorem 7 (Berezansky). *Almost every E with respect to the spectral measure admits a generalized eigenfunction of polynomial growth.*

In particular, Theorem 7 implies that for any potential v, for any $\omega \in \Omega$, almost every E with respect to the spectral measure of the Schrödinger operator associated to ω, there exists $X \in \mathbb{R}^2$, $C, d > 0$ such that

$$\|A^E(n, \omega)X\| \le C(|n| + 1)^d, \forall n \in \mathbb{Z}$$

2.1. **Notations.** Throughout this paper, we will use $\lesssim$ and $\gtrsim$ to denote less than or greater than up to multiplying a universal constant. In places we use Laudau's $O(f)$ to denote a quantity majorized by a universal constant times f, and use $\Theta(f)$ to denote a quantity minorized by a positive universal constant times f.

For any $a, b \in \mathbb{R}$, we will use $|a - b|_{\mathbb{R}/\pi\mathbb{Z}}$ to denote the distance from $a - b$ to the set $\{k\pi\}_{k\in\mathbb{Z}}$. For any two vectors $X_1, X_2 \in \mathbb{R}^2$ such that $X_i = r_i \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}$ for $i = 1, 2$, we denote $\angle(X_1, X_2) = |\theta_1 - \theta_2|_{\mathbb{R}/\pi\mathbb{Z}}$.

3. A TOWER CONSTRUCTION

In order to prove Theorem 3, it suffices to prove that for any $\alpha \in \mathcal{A}$, we have the corresponding upper bound for the Lebesgue measure of $\Sigma \cap I_{v(\alpha)}$. Then Theorem 3 will follow from (2.2) and the fact that $card(A) < \infty$.

Throughout Section 3 to Section 7, we fix $\alpha_0 \in \mathcal{A}$ and denote $E_0 = v(\alpha_0) \in R$. Then Theorem 3 is reduced to the following.

Theorem 8. *Under the condition of Theorem 3, for any $0 < \gamma < \frac{1}{4}$ where L is given by condition (2) in Theorem 3, there exists a constant $Q > 0$, such that for any admissible potential $v : \{1, \cdots, k\} \rightarrow \mathbb{R}$, we have $Leb(\Sigma_v \cap I_{E_0}) < Q\lambda_v^{-\gamma}$.*

Hereafter, we will assume that the condition in Theorem 3 holds. We denote

$$\lambda = \lambda_v$$

Define

$$\Delta = \{\omega \in \Omega; \omega_{-1} \neq \alpha_0, \omega_0 = \alpha_0\}$$
$$\Delta_{(i)} = \{\omega \in \Omega; \omega_{-1} \neq \alpha_0, \omega_0 = \alpha_0, \cdots, \omega_{i-1} = \alpha_0, \omega_i \neq \alpha_0\}$$

Since ergodic subshift (Ω, T, μ) satisfies the condition (1) in Theorem 3, then there exists $K > 0$ such that

$$\Delta = \bigsqcup_{i=1}^{K} \Delta_{(i)} \text{ up to a } \mu-\text{null set}$$

We define

$$\Delta_0 = \Delta$$

By our assumptions in Section 2, we have

$$\mu(\Delta_0) > 0$$

After possibly removing a $\mu-$null set from Δ_0, we can assume that for any $\omega \in \Delta_0$, there exist integers $n, m > 0$ such that $T^n(\omega) \in \Delta_0$ and $T^{-m}(\omega) \in \Delta_0$.

For any $E \in \mathbb{R}$, for any $\omega \in \Delta_0$, we define

$$l_0(\omega) = \inf\{k; k > 0, T^k(\omega) \in \Delta_0\}$$
$$T_0(\omega) = T^{l_0(\omega)}(\omega)$$
$$A_0^E(\omega) = A^E(l_0(\omega), \omega)$$

Note that there is an ergodic T_0-invariant probability measure μ_0 on Δ_0 given by

$$\mu_0 = \frac{1}{\Delta_0}\mu|_{\Delta_0}$$

For any $E \in \mathbb{R}$, we denote $C^E = A_{\alpha_0}^E$ and $C_i^E = (C^E)^i$. For any $\omega \in \Delta_{(i)}$, we have that

$$A^E(T^k\omega) = C^E \text{ for all } 0 \leq k \leq i-1$$

For any $1 \leq i \leq K$, for all $\omega \in \Delta_{(i)}$, we define

$$
\begin{aligned}
C^E(\omega) &= A^E(i, \omega) = C_i^E \\
B_0^E(\omega) &= A^E(l_0(\omega) - i, T^i(\omega)) = A_0^E(\omega)(C^E(\omega))^{-1}
\end{aligned}
$$

In the following, for any $n \geq 0$, we are going to define $\Delta_n \subset \Delta_0$, to which we associate a map $T_n : \Delta_n \to \Delta_n$, an ergodic T_n-invariant probability measure μ_n, functions $l_n : \Delta_n \to \mathbb{Z}$, $r_n : \Delta_{n+1} \to \mathbb{Z}$, $A_n^E, B_n^E : \Delta_n \to SL(2, \mathbb{R})$ satisfying the following properties:

(P1) For any $n \geq 0$, $\mu(\Delta_n) > 0$ and $\Delta_{n+1} \subset \Delta_n$;

(P2) For every $\omega \in \Delta_n$, $l_n(\omega) = \inf\{m > 0; T^m(\omega) \in \Delta_n\} < \infty$ and $T_n(\omega) = T^{l_n(\omega)}(\omega)$;

(P3) $\mu_n = \frac{1}{\mu(\Delta_n)} \mu|_{\Delta_n}$;

(P4) $A_n^E(\omega) = A^E(l_n(\omega), \omega)$;

(P5) For each $1 \leq i \leq K$, for any $\omega \in \Delta_n \cap \Delta_{(i)}$, we have $A_n^E(\omega) = B_n^E(\omega) C_i^E$.

(P6) $r_n(\omega) = \inf\{k; k > 0, T_n^k(\omega) \in \Delta_{n+1}\} < \infty$ for all $\omega \in \Delta_{n+1}$.

By (P2), we see that T_n is the Poincaré return map on Δ_n.

By (P2),(P4),(P5) and (P6) we get

$$(3.1)\ l_{n+1}(\omega) = \sum_{i=0}^{r_n(\omega)-1} l_n(T_n^i \omega), \forall n \geq 0, \forall \omega \in \Delta_{n+1}$$

$$(3.2)\ \ B_n^E(\omega) = A^E(l_n(\omega) - i, T^i(\omega)), \forall n \geq 0, \forall \omega \in \Delta_n \bigcap \Delta_{(i)}$$

By the definition of $\Delta_0, \Delta_{(i)}$, we see that $l_0(\omega) \geq 1$ for all $\omega \in \Delta_0$. Hence by (3.1), we have $l_n(\omega) \geq 1$ for all $n \geq 0$ and $\omega \in \Delta_n$.

NOTATION 1. *For any matrix $A \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$, we denote $u(A), s(A), \lambda(A)$ to be real numbers that satisfy*

$$
A = R_{u(A)} \begin{bmatrix} \lambda(A) & 0 \\ 0 & \lambda(A)^{-1} \end{bmatrix} R_{\frac{\pi}{2} - s(A)}
$$

and $\lambda(A) > 1$

Here $u(A), s(A)$ are well-defined up to adding a multiple of π.

In Section 5 we will construct a finite union of intervals, denoted by $J_n \subset I_{E_0}$ for each $n \geq 0$. We now introduce a sequence of parameters $\bar{\lambda}_n, \zeta_n, \chi_n, M_n, N_n, \kappa_n > 0$ satisfying the following estimates:

$$(3.3) \quad 0 < \sup_{\omega \in \Delta_n} l_n(\omega) \leq M_n \inf_{\omega \in \Delta_n} l_n(\omega)$$

$$(3.4) \quad r_n(\omega) \in \{N_n, N_n + 1\}, \forall n \geq 0, \forall \omega \in \Delta_{n+1}$$

$$(3.5) \quad \sum_{n=0}^{\infty} \frac{1}{N_n} < \infty$$

For any $n \geq 0$, any $E \in I_{E_0} \setminus \bigcup_{n \geq m \geq 0} J_m$, any $\omega \in \Delta_{n+1}$, any $0 \leq q < r \leq r_n(\omega)$, denote

$$B^E = A_n^E(T_n^{r-1}(\omega)) \cdots A_n^E(T_n^{q+1}(\omega)) B_n^E(T_n^q(\omega))$$

then

$$(3.6) \quad B^E \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$$

and

$$(3.7) \quad |u(B^E) - u(B_n^E(T_n^{r-1}(\omega)))|_{\mathbb{R}/\pi\mathbb{Z}} \leq \zeta_n$$

$$(3.8) \quad |s(B^E) - s(B_n^E(T_n^q(\omega)))|_{\mathbb{R}/\pi\mathbb{Z}} \leq \zeta_n$$

$$(3.9) \quad \lambda(B^E) \geq e^{\chi_{n+1} \sum_{i=q}^{r-1} l_n(T_n^i(\omega))}$$

Note that by taking $r = q + 1$ and (3.6), we have $B_n^E(T_n^{r-1}(\omega)), B_n^E(T_n^q(\omega)) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$. This shows that the left hand side of (3.7) and (3.8) are well-defined.

Moreover, for any $n \geq 0$, any $E \in I_{E_0} \setminus \bigcup_{n-1 \geq m \geq 0} J_m$, any $\omega \in \Delta_n$, we have

$$(3.10) \quad \lambda(B_n^E(\omega)) \geq e^{\chi_n l_n(\omega)} \geq \bar{\lambda}_n$$

We will choose an absolute constant $C > 1$ such that

$$(3.11) \quad \|C_k^E\|, \|\partial_E C_k^E\| \leq C, \forall 1 \leq k \leq K, \forall E \in I_{E_0}$$

We will use the following lemma to determine the values of $\bar{\lambda}_0, \chi_0, M_0$.

LEMMA 1. *We can choose* $M_0 = \dfrac{\sup_{\omega \in \Delta_0} l_0(\omega)}{\inf_{\omega \in \Delta_0} l_0(\omega)} < \infty$ *so that (3.3) is valid for* $n = 0$. *For any* $\lambda > 0$ *sufficiently large, we can choose* $\bar{\lambda}_0 = \frac{1}{2}\lambda, \chi_0 = \log \bar{\lambda}_0$ *so that (3.10) is valid for* $n = 0$. *Moreover, for any* $\lambda > 0$ *sufficiently large, we have (3.6) for* $n = 0$.

Proof. The hypothesis (3.3) follows from the definition of M_0. It follows from condition (2) in Theorem 3 that $M_0 < \infty$.

For all λ sufficient large, for any $E \in I_{E_0}$, any $\alpha \in \mathcal{A}$ distinct from α_0, $A_\alpha^E = \begin{bmatrix} \eta & -1 \\ 1 & 0 \end{bmatrix}$ with $\eta = E - v(\alpha)$. When λ is sufficiently large, we have $|E - v(\alpha)| \geq |v(\alpha_0) - v(\alpha)| - |v(\alpha_0) - E| \geq \lambda - H > \frac{9}{10}\lambda$. It is direct to check that there exists absolute constants $\epsilon > 0$, $\Lambda > 0$, such that the following is true. Denote $\mathcal{C} \subset \mathbb{R}^2 \setminus \{(0,0)\}$ as

$$\mathcal{C} = \{(x,y); x \neq 0, |y| \leq \epsilon|x|\}$$

for any $X \in \mathcal{C}$, for any η such that $|\eta| > \Lambda$ we have

$$\begin{bmatrix} \eta & -1 \\ 1 & 0 \end{bmatrix} X \in \mathcal{C} \text{ and} \left\| \begin{bmatrix} \eta & -1 \\ 1 & 0 \end{bmatrix} X \right\| \geq \frac{2}{3}\eta \|X\|$$

Then when λ is sufficiently large, for any $\alpha \in \mathcal{A}$ distinct from α_0, any $E \in I_{E_0}$ and any $X \in \mathcal{C}$, we have

$$A_\alpha^E X \in \mathcal{C} \text{ and} \|A_\alpha^E X\| \geq \frac{9}{10} \times \frac{2}{3}\lambda \|X\| \geq \frac{1}{2}\lambda \|X\|$$

Since for any $\omega \in \Delta_0$, B^E in (3.6) is a product of some matrices in set $\{A_\alpha^E\}_{\alpha \neq \alpha_0}$, we have (3.10), (3.6) for $n = 0$ with our choices of $\bar{\lambda}_0, \chi_0$ in the statement. This completes the proof. $\qquad\square$

The sets J_n will be defined and the precise choices of parameters $\bar{\lambda}_n, \zeta_n, \chi_n, M_n, N_n, \kappa_n$ will be made clear in Section 5.

We have the following lemma that will be used repeatedly.

LEMMA 2. *There exists $c_5 > 0$ such that for any $\epsilon > 0$, any $u, s, \tilde{u}, \tilde{s} \in \mathbb{R}$ satisfying $|u - \tilde{u}|_{\mathbb{R}/\pi\mathbb{Z}}, |s - \tilde{s}|_{\mathbb{R}/\pi\mathbb{Z}} < \epsilon$, for any $1 \leq k \leq K$, any $E \in I_{E_0}$, we have*

$$\angle \left(R_{\frac{\pi}{2} - \tilde{s}} C_k^E R_{\tilde{u}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, R_{\frac{\pi}{2} - s} C_k^E R_u \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) < c_5 \epsilon$$

Proof. Since the norm of C_k^E is uniformly bounded for all $E \in I_{E_0}$ and $1 \leq k \leq K$, the lemma follows from straight-forward calculations. $\qquad\square$

4. AN ITERATION SCHEME

In this section, we will prove a lemma that will help us control the dynamics for energies that satisfy certain transversality condition. Throughout this section, we will use the following notations.

NOTATION 2. *For any $E \in I_{E_0}$, any $n \geq 0$, any $\omega \in \Delta_n$, integer $r \geq 1$ such that*

(4.1) $B_n^E(T_n^j(\omega)) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R}), \forall 0 \leq j \leq r - 1$

we denote

$$u_j(E) = u(B_n^E(T_n^j(\omega))), s_j(E) = s(B_n^E(T_n^j(\omega))), \lambda_j(E) = \lambda(B_n^E(T_n^j(\omega)))$$

for all $0 \leq j \leq r - 1$.

Denote

$$
\begin{aligned}
C^{E,j} &= C^E(T_n^j(\omega)) \\
B^E &= A_n^E(T_n^{r-1}(\omega)) \cdots A_n^E(T_n(\omega)) B_n^E(\omega) \\
&= B_n^E(T_n^{r-1}(\omega)) C^{E,r-1} \cdots C^{E,2} B_n^E(T_n(\omega)) C^{E,1} B_n^E(\omega) \\
D_j(E) &= R_{\frac{\pi}{2} - s_{j+1}(E)} C^{E,j+1} R_{u_j(E)}
\end{aligned}
$$

for all $0 \leq j \leq r - 2$.

By (P2),(P5) and (P6), when $\omega \in \Delta_{n+1}$ and $r = r_n(\omega)$ we have

$$B^E = B_{n+1}^E(\omega)$$

The main goal of this section is the following lemma, which says that under certain transversality conditions, we can give good lower bound for the norm of B^E when r is not too large, and at the same time, keep track of its stable,unstable directions. Similar estimates can be found in [2], [11], [12]. We need a slightly more precise estimate. We should notice that we only require the C^0 norm control of the stable/unstable directions. In this aspect, our lemma is simpler than the ones in those papers mentioned above.

LEMMA 3. *There exists a constant $\Lambda > 0$ and $C_2, P > 0$ such that the following is true. For any $n \geq 0$, $r \geq 1$, $E \in I_{E_0}$, any $\omega \in \Delta_n$ such that (4.1) holds. Let $B^E, u_j(E), s_j(E), \lambda_j(E)$ for $0 \leq j \leq r - 1$, $D_j(E)$ for $0 \leq j \leq r - 2$ be defined in Notation 2. Assume that we have*

$$(4.2) \qquad\qquad \lambda_l(E) > \bar{\lambda}_n > \Lambda, \forall 0 \leq l \leq r - 1$$

$$(4.3) \qquad \angle(D_l(E) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) > \kappa_n > 2\bar{\lambda}_n^{-\frac{1}{4}}, \forall 0 \leq l \leq r - 2$$

$$(4.4) \qquad\qquad\qquad r < C_2^{-1} \kappa_n^3 \bar{\lambda}_n^2$$

Then $B^E \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$. Moreover,

$$\lambda(B^E) \geq C^{-Pr} \prod_{l=0}^{r-1} \lambda_l(E) \kappa_n^r$$

$$|s(B^E) - s_0(E)|_{\mathbb{R}/\pi\mathbb{Z}}, |u(B^E) - u_{r-1}(E)|_{\mathbb{R}/\pi\mathbb{Z}} \leq C^P \bar{\lambda}_n^{-2} \kappa_n^{-2}(r-1)$$

The key ingredient in the proof of Lemma 3 is the following lemma, which corresponds to the statement in Lemma 3 when $r = 2$.

LEMMA 4. *For any $C_0 \geq 1$, there exists $\hat{\lambda} > C_0$, such that for any $\bar{\lambda} > \hat{\lambda}$, $\kappa \in (0,1)$, for $\lambda_0, \lambda_1 > 0$, $D \in SL(2, \mathbb{R})$, let*

$$(4.5) \qquad A \; = \; \begin{bmatrix} \lambda_1 & \\ & \lambda_1^{-1} \end{bmatrix} D \begin{bmatrix} \lambda_0 & \\ & \lambda_0^{-1} \end{bmatrix}$$

If we have

$$(4.6) \qquad \qquad \|D\| \; \leq \; C_0$$
$$(4.7) \qquad \qquad \min(\lambda_0, \lambda_1) \; \geq \; \bar{\lambda}$$
$$(4.8) \qquad \qquad \angle(D \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \; > \; \kappa > \bar{\lambda}^{-\frac{1}{4}}$$

then we have $A \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$. Moreover,

$$(4.9) \qquad \qquad \lambda(A) \; \gtrsim \; C_0^{-1} \lambda_0 \lambda_1 \kappa$$
$$(4.10) \qquad |\frac{\pi}{2} - s(A)|_{\mathbb{R}/\pi\mathbb{Z}}, |u(A)|_{\mathbb{R}/\pi\mathbb{Z}} \; \lesssim \; C_0^{O(1)} \bar{\lambda}^{-2} \kappa^{-2}$$

Proof of Lemma 4. We denote

$$f(x) = \|A \begin{bmatrix} \cos x \\ \sin x \end{bmatrix} \|^2$$

and

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

By (4.6), we have that

$$(4.11) \qquad \qquad |a|, |b|, |c|, |d| \leq C_0$$

By (4.8), (4.6) we have

$$(4.12) \qquad |a| \; = \; |\langle D \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle|$$

$$\geq \; \frac{1}{\|D\|} \sin \angle(D \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) > \frac{1}{10} C_0^{-1} \kappa$$

Simple calculations show that

$$(4.13) \qquad \qquad A \; = \; \begin{bmatrix} \lambda_0 \lambda_1 a & \lambda_0^{-1} \lambda_1 b \\ \lambda_1^{-1} \lambda_0 c & \lambda_0^{-1} \lambda_1^{-1} d \end{bmatrix}$$

Then

$$A \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} \; = \; \begin{bmatrix} \lambda_0 \lambda_1 a \cos \alpha + \lambda_0^{-1} \lambda_1 b \sin \alpha \\ \lambda_1^{-1} \lambda_0 c \cos \alpha + \lambda_0^{-1} \lambda_1^{-1} d \sin \alpha \end{bmatrix}$$

and

(4.14) $f(\alpha) = (\lambda_0 \lambda_1 a \cos \alpha + \lambda_0^{-1} \lambda_1 b \sin \alpha)^2 +$
$$(\lambda_1^{-1} \lambda_0 c \cos \alpha + \lambda_0^{-1} \lambda_1^{-1} d \sin \alpha)^2$$

Note that by (4.13), (4.6), (4.7), (4.12) and the second inequality in (4.8), we have

$$(4.15) \qquad \begin{aligned} tr(A) &= \lambda_0 \lambda_1 a + \lambda_0^{-1} \lambda_1^{-1} d \\ &\geq \frac{1}{10 C_0} \bar{\lambda}^2 \kappa - \bar{\lambda}^{-2} C_0 \\ &\geq \frac{1}{10 C_0} \hat{\lambda} - \hat{\lambda}^{-2} C_0 \\ &> 2 \end{aligned}$$

when $\hat{\lambda}$ is bigger than some constant depending only on C_0. Thus $A \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$ when $\hat{\lambda}$ is sufficiently large depending only on C_0.

Define θ by setting

$$f(\theta) = \sup_{\alpha} f(\alpha)$$

Note that by (4.15), θ is uniquely defined up to a multiple of π.

Then

$$(4.16) \qquad\qquad f'(\theta) = 0$$

We have

$$(4.17) \qquad f'(\alpha) = 2\cos(2\alpha) L_1 + \sin(2\alpha) L_2$$

where

$$(4.18) \quad L_1 = \lambda_1^2 ab + \lambda_1^{-2} cd$$
$$(4.19) \quad L_2 = -\lambda_0^2 \lambda_1^2 a^2 + \lambda_0^{-2} \lambda_1^2 b^2 - \lambda_0^2 \lambda_1^{-2} c^2 + \lambda_0^{-2} \lambda_1^{-2} d^2$$

By (4.16),(4.17) we have either

$$\theta = \frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2}) \mod \pi\mathbb{Z}$$

$$\text{or} \qquad \theta = \frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2}) + \frac{\pi}{2} \mod \pi\mathbb{Z}$$

here we consider function tan as a function from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$.

Now we estimate $|L_1|, |L_2|$.

By (4.11) and (4.18), we have

$$(4.20) \qquad\qquad |L_1| < 2\lambda_1^2 C_0^2$$

When $\bar{\lambda}$ is sufficiently large depending only on C_0, by (4.12),(4.8) and (4.19) we have

$$(4.21) \qquad C_0^2 \lambda_0^2 \lambda_1^2 \gtrsim |L_2| \gtrsim C_0^{-2} \lambda_0^2 \lambda_1^2 \kappa^2$$

Hence by (4.20) and (4.21)

$$(4.22) \qquad |\frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2})| \lesssim |\frac{L_1}{L_2}| \lesssim C_0^4 \lambda_0^{-2} \kappa^{-2}$$

Now we are going to compare $f(\frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2}))$ and $f(\frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2}) + \frac{\pi}{2})$.

If $\theta = \frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2})$, when $\hat{\lambda}$ is bigger than some constant depending only on C_0, by (4.14), (4.17), (4.12), (4.7) ,(4.11) and the second inequality in (4.8) we have that

$$\begin{aligned} f(\theta) &\gtrsim (\lambda_0 \lambda_1 C_0^{-O(1)} \kappa - \lambda_0^{-1} \lambda_1 C_0)^2 - C_0^2 (\lambda_1^{-1} \lambda_0 + \lambda_0^{-1} \lambda_1^{-1})^2 \\ &\gtrsim C^{-O(1)} \lambda_0^2 \lambda_1^2 \kappa^2 \gtrsim C^{-O(1)} \max(\lambda_0^2, \lambda_1^2) \bar{\lambda}^{\frac{3}{2}} \end{aligned}$$

If $\theta = \frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2}) + \frac{\pi}{2}$, when $\hat{\lambda} > 1$, by (4.14),(4.17), (4.7), (4.11), (4.22) and the second inequality in (4.8) we have that

$$\begin{aligned} f(\theta) &\lesssim (\lambda_0^{-1} \lambda_1 C_0^{O(1)} \kappa^{-2} + \lambda_0^{-1} \lambda_1 C_0)^2 + C_0^2 (\lambda_1^{-1} \lambda_0 + \lambda_0^{-1} \lambda_1^{-1})^2 \\ &\lesssim C^{O(1)} (\lambda_1^2 + \lambda_0^2) \kappa^{-4} \lesssim C^{O(1)} \max(\lambda_1^2, \lambda_0^2) \bar{\lambda} \end{aligned}$$

Thus we have showed that $f(\frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2})) > f(\frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2}) + \frac{\pi}{2})$ when $\bar{\lambda}$ is sufficiently large depending only on C_0. Since clearly that f is $\pi-$periodic, this implies that we can take

$$(4.23) \qquad \theta = \frac{1}{2} \tan^{-1}(-\frac{2L_1}{L_2})$$

Since $\bar{\lambda} > \hat{\lambda}$, when $\hat{\lambda}$ is sufficiently large depending only on C_0, by (4.23) and (4.22) we have

$$|\theta| \lesssim C_0^4 \lambda_0^{-2} \kappa^{-2}$$

By definition, we have $\theta = s - \frac{\pi}{2}$ modulo $\pi \mathbb{Z}$. Thus we have

$$|\frac{\pi}{2} - s|_{\mathbb{R}/\pi\mathbb{Z}} \lesssim C_0^4 \lambda_0^{-2} \kappa^{-2} < C_0^4 \bar{\lambda}^{-2} \kappa^{-2}$$

By symmetry, we have

$$|u|_{\mathbb{R}/\pi\mathbb{Z}} \lesssim C_0^4 \bar{\lambda}^{-2} \kappa^{-2}$$

This proves (4.10).

By (4.6), (4.7) and (4.8),

$$f \gtrsim C_0^{-2} \lambda_0^2 \lambda_1^2 \kappa^2$$

It is easy to see that

$$\sigma = f(\theta)^{\frac{1}{2}}$$

Hence

$$\sigma \gtrsim C_0^{-1}\lambda_0\lambda_1\kappa$$

This proves (4.9).

$\square$

Proof of Lemma 3. Recall that $B^E, u_k(E), s_k(E), \lambda_k(E)$ are defined in Notation 2. By definition

$$B_n^E(T_n^j(\omega)) \;=\; R_{u_j(E)} \begin{bmatrix} \lambda_j(E) & 0 \\ 0 & \lambda_j(E)^{-1} \end{bmatrix} R_{\frac{\pi}{2}-s_j(E)}, \forall 0 \leq j \leq r-1$$

Then

$$B^E = R_{u_{r-1}(E)} \begin{bmatrix} \lambda_{r-1}(E) & \\ & \lambda_{r-1}(E)^{-1} \end{bmatrix} D_{r-2} \begin{bmatrix} \lambda_{r-1}(E) & \\ & \lambda_{r-1}(E)^{-1} \end{bmatrix} \cdots$$

$$D_1 \begin{bmatrix} \lambda_1(E) & \\ & \lambda_1(E)^{-1} \end{bmatrix} D_0 \begin{bmatrix} \lambda_0(E) & \\ & \lambda_0(E)^{-1} \end{bmatrix} R_{\frac{\pi}{2}-s_0(E)}$$

For all $l \geq 0$, we denote

$$B^{(l),E} = \begin{bmatrix} \lambda_l(E) & \\ & \lambda_l(E)^{-1} \end{bmatrix} D_{l-1} \begin{bmatrix} \lambda_{l-1}(E) & \\ & \lambda_{l-1}(E)^{-1} \end{bmatrix} \cdots$$

$$D_1 \begin{bmatrix} \lambda_1(E) & \\ & \lambda_1(E)^{-1} \end{bmatrix} D_0 \begin{bmatrix} \lambda_0(E) & \\ & \lambda_0(E)^{-1} \end{bmatrix}$$

In particular, we have

$$B^{(0),E} \;=\; \begin{bmatrix} \lambda_0(E) & \\ & \lambda_0(E)^{-1} \end{bmatrix}$$

For any l such that $B^{(l),E} \in SL(2,\mathbb{R}) \setminus SO(2,\mathbb{R})$, we denote functions

$$u_{(l)} = u(B^{(l),E}), \quad s_{(l)} = s(B^{(l),E}), \quad \sigma_l = \lambda(B^{(l),E})$$

where

(4.24) $$u_{(0)} = 0, \quad s_{(0)} = \frac{\pi}{2}, \quad \sigma_0 = \lambda_0$$

We have

(4.25) $$B^{(l+1),E} \;=\; \begin{bmatrix} \lambda_{l+1}(E) & \\ & \lambda_{l+1}(E)^{-1} \end{bmatrix} D_l(E) R_{u_{(l)}(E)}$$

$$\cdot \begin{bmatrix} \sigma_l(E) & \\ & \sigma_l(E)^{-1} \end{bmatrix} R_{\frac{\pi}{2}-s_{(l)}(E)}$$

(4.26) $$B^E \;=\; R_{u_{r-1}(E)} B^{(r-1),E} R_{\frac{\pi}{2}-s_0(E)}$$

We will inductively show that for some absolute constant $P > 0$, for all $0 \leq l \leq r - 1$ we have,

$$(4.27) \qquad\qquad\qquad\qquad \sigma_l \;>\; \bar{\lambda}_n$$

$$(4.28) \qquad |u_{(l)}|_{\mathbb{R}/\pi\mathbb{Z}}, |\tfrac{\pi}{2} - s_{(l)}|_{\mathbb{R}/\pi\mathbb{Z}} \;\leq\; C^P l \kappa_n^{-2} \bar{\lambda}_n^{-2}$$

By (4.24), we clearly have (4.27), (4.28) for $l = 0$.

Assume that for some $0 \leq l \leq r - 2$, (4.27) and (4.28) are valid and $B^{(l),E} \in SL(2,\mathbb{R}) \setminus SO(2,\mathbb{R})$. By (4.28) for l, we apply Lemma 2, (4.4) and (4.3) to see that

$$\angle(D_l(E)R_{u_{(l)}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \;>\; \angle(D_l(E) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) - c_5 C^{O(1)} l \kappa_n^{-2} \bar{\lambda}_n^{-2}$$

$$>\; \kappa_n - c_5 C^{O(1)} r \kappa_n^{-2} \bar{\lambda}_n^{-2}$$

$$(4.29) \qquad\qquad\qquad\qquad\qquad >\; \frac{1}{2} \kappa_n$$

The last inequality is true by (4.4) when we choose C_2 to be sufficiently large. We note that C_2 can be taken to be an absolute constant.

By (4.2), we have

$$\lambda_{l+1}(E) \;>\; \bar{\lambda}_n$$

When $\Lambda > \hat{\lambda}$ where $\hat{\lambda}$ is given by Lemma 4 with $C_0 = C$, we apply Lemma 4 for $\lambda_1 = \lambda_{l+1}(E), \lambda_0 = \sigma_l(E), D = D_l(E)R_{u_{(l)}}$. We note that by (4.27) for l, (4.29), (4.3) and (4.2) that the condition of Lemma 4 is satisfied for $\kappa = \frac{1}{2}\kappa_n$.

By Lemma 4, we have

$$\begin{bmatrix} \lambda_{l+1}(E) & \\ & \lambda_{l+1}(E)^{-1} \end{bmatrix} D \begin{bmatrix} \sigma_l(E) & \\ & \sigma_l(E)^{-1} \end{bmatrix} \in SL(2,\mathbb{R}) \setminus SO(2,\mathbb{R})$$

By (4.25), we obtain that $B^{(l+1),E} \in SL(2,\mathbb{R}) \setminus SO(2,\mathbb{R})$, Moreover,

$$\begin{bmatrix} \lambda_{l+1}(E) & \\ & \lambda_{l+1}(E)^{-1} \end{bmatrix} D_l(E) R_{u_{(l)}(E)} \begin{bmatrix} \sigma_l(E) & \\ & \sigma_l(E)^{-1} \end{bmatrix}$$

$$=\; R_{u_{(l+1)}} \begin{bmatrix} \sigma_{l+1}(E) & \\ & \sigma_{l+1}(E)^{-1} \end{bmatrix} R_{s_{(l)} - s_{(l+1)}}$$

Then by Lemma 4 and the fact we assumed $C > 1$, we see that by enlarging P if necessary, we obtain

$$(4.30) \qquad |u_{(l+1)}|_{\mathbb{R}/\pi\mathbb{Z}}, |s_{(l)} - s_{(l+1)}|_{\mathbb{R}/\pi\mathbb{Z}} \;\leq\; C^P \bar{\lambda}_n^{-2} \kappa_n^{-2}$$

$$(4.31) \qquad\qquad\qquad\qquad\qquad \sigma_{l+1} \;\geq\; C^{-P} \lambda_{l+1} \sigma_l \kappa_n$$

We note that we can choose P to be an absolute constant.

By (4.28), (4.30), we have

$$|u_{(l+1)}|_{\mathbb{R}/\pi\mathbb{Z}}, |\frac{\pi}{2} - s_{(l+1)}|_{\mathbb{R}/\pi\mathbb{Z}} \leq C^P(l+1)\kappa_n^{-2}\bar{\lambda}_n^{-2}$$

This recoved estimate (4.28) for $l+1$.

Since by (4.27) and the second inequality in (4.3), we see that

$$\sigma_l\kappa_n > \bar{\lambda}_n^{\frac{1}{2}} \geq \Lambda^{\frac{1}{2}}$$

Then by (4.31) and (4.2) we have

$$\sigma_{l+1} > \bar{\lambda}_n$$

when Λ is sufficiently large depending only on C. Hence we have recovered estimates (4.27) for $l+1$ and have completed the induction. Moreover, we see that (4.31) holds for any $0 \leq l \leq r-2$.

By (4.27) for $l = r-1$, we get

$$(4.32) \qquad |u_{(r-1)}(E)|_{\mathbb{R}/\pi\mathbb{Z}} \leq C^P\bar{\lambda}_n^{-2}\kappa_n^{-2}(r-1)$$

$$(4.33) \qquad |\frac{\pi}{2} - s_{(r-1)}(E)|_{\mathbb{R}/\pi\mathbb{Z}} \leq C^P\bar{\lambda}_n^{-2}\kappa_n^{-2}(r-1)$$

Concatenating the estimates (4.31) for $0 \leq l \leq r-2$, and using $C > 1$, we get

$$(4.34) \qquad \sigma_{r-1}(E) > C^{Pr}\kappa_n^{r-1}\prod_{i=0}^{r-1}\lambda_i(E)$$

By (4.26), we have that

$$\lambda(B^E) = \sigma_{r-1}$$
$$u(B^E) = u_{r-1} + u_{(r-1)}$$
$$s(B^E) = s_0 + s_{(r-1)} - \frac{\pi}{2}$$

Then the lemma follows from (4.32), (4.33) and (4.34) $\qquad\square$

5. CHOOSING THE PARAMETERS

In this section, we will introduce several sets that will help us estimate the area of the spectrum in Section 6, 7.

Definition 3. *For any $n \geq 0$, we define*

$$\mathcal{A}_n = \bigcup_{i=1}^{K}\{\omega_i\omega_{i+1}\cdots\omega_{l_n(\omega)-1}; \omega \in \Delta_n \bigcap \Delta_{(i)}\}$$

For $\alpha, \beta \in \mathcal{A}_n, 1 \leq j \leq K, \epsilon > 0$, we define $J(\alpha, \beta, j, \epsilon) \subset I_{E_0}$ as

$$J(\alpha, \beta, j, \epsilon) = \{E \in I_{E_0}; A^E(\alpha), A^E(\beta) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R}) \text{ and}$$

$$\angle(R_{\frac{\pi}{2} - s(A^E(\beta))} C_j^E R_{u(A^E(\alpha))} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \leq \epsilon\}$$

For a given choice of the sequence $\{\Delta_n\}_{n \in \mathbb{N}}$ (which in turn determines $\{A_n^E\}_{n \in \mathbb{N}}, \{B_n^E\}_{n \in \mathbb{N}},$ etc.) and $\{\kappa_n\}_{n \in \mathbb{N}}$, we define

(5.1) $J_n = \bigcup_{\alpha \in \mathcal{A}_n, \beta \in \mathcal{A}_n, 1 \leq j \leq K} J(\alpha, \beta, j, \kappa_n)$

(5.2) $J = \bigcup_n J_n$

By Definition 3, (P5) we see that, for any $n \geq 0, 1 \leq j \leq K$, any $\omega, \tilde{\omega} \in \Delta_n$, any $E \in J_n$, we have

(5.3) $B_n^E(\omega), B_n^E(\tilde{\omega}) \in SL(2, \mathbb{R}) \quad \setminus \quad SO(2, \mathbb{R})$

(5.4)   and $\angle(R_{\frac{\pi}{2} - s(B_n^E(\omega))} C_j^E R_{u(B_n^E(\tilde{\omega}))} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \quad \leq \quad \kappa_n$

Now we will choose the parameters $\bar{\lambda}_n, \zeta_n, \chi_n, M_n, N_n, \kappa_n$ which were introduced in Section 3. In the rest of this paper, we use the following notation.

NOTATION 3. *For any $n \geq 0$, we denote*

$$\inf l_n = \inf_{\omega \in \Delta_n} l_n(\omega)$$

$$\sup l_n = \sup_{\omega \in \Delta_n} l_n(\omega)$$

The goal of this section is to show the following lemma.

LEMMA 5. *For any $0 < \gamma < \gamma' < \frac{1}{4}$, any $0 < c < 2 - 3\gamma'$, there exists $C', C'', \Gamma > 0$ such that the following is true. For any admissible potential v, denote $\lambda = \lambda_v$, such that $\lambda > \Gamma$, then there exists $\{\Delta_n\}_{n \in \mathbb{N}}$, and parameters $\bar{\lambda}_n, \zeta_n, \chi_n, M_n, N_n, \kappa_n$ such that :*

Let $\bar{\lambda}_0, \chi_0, M_0$ be given by Lemma 1. For any $n \geq 0$, we define J_n by (5.1). Then we have (3.3) to (3.10). Moreover, for all $n \geq 0$ we have

(5.5) $\zeta_n < \bar{\lambda}_n^{-c} \leq \bar{\lambda}_0^{-2^n c}$

(5.6) $\chi_n > C' \chi_0$

(5.7) $\bar{\lambda}_n^{-\gamma'} < \kappa_n < \bar{\lambda}_0^{-\gamma}$

(5.8) $M_n \leq C'' M_0$

Proof. Denote

$$(5.9) \qquad \xi = -\frac{1}{10} \log \gamma'$$

By the condition $\gamma' < 1$, we have $\xi > 0$.

We choose an arbitrary sequence of integers $\{N_n\}$, such that

$$(5.10) \qquad \sum_{n=0}^{\infty} \log \frac{N_n + 1}{N_n} \quad < \quad \xi$$

$$(5.11) \qquad \frac{2}{1 - e^{-\xi}} \quad \leq \quad N_n, \forall n \geq 0$$

$$(5.12) \qquad N_{n+1} \quad \leq \quad 2N_n, \forall n \geq 0$$

By (5.10), we get (3.5).

Assume that Δ_m is defined for all $0 \leq m \leq n$ for some $n \geq 0$ (Δ_0 is defined in Section 3). By Rokhlin tower theorem and aperiodicity, we can and do choose

$$\Delta_{n+1} \subset \Delta_n$$

such that for any $\omega \in \Delta_{n+1}$, we have

$$r_n(\omega) \in \{N_n, N_n + 1\}$$

We inductively define Δ_n for all $n \geq 0$ and we get (3.4) for all $n \geq 0$.

We define that

$$(5.13) \qquad M_{n+1} = \frac{N_n + 1}{N_n} M_n, \forall n \geq 0$$

where M_0 is defined in Lemma 1 as $M_0 = \frac{\sup l_0}{\inf l_0}$. Since we already showed (3.4), by (3.1) and (3.4) we have for any $n \geq 0$, for any $\omega \in \Delta_{n+1}$,

$$l_{n+1}(\omega) \leq r_n(\omega) \sup l_n \leq (N_n + 1) \sup l_n$$

and

$$l_{n+1}(\omega) \geq r_n(\omega) \inf l_n \geq N_n \inf l_n$$

If we have $\sup l_n \leq M_n \inf l_n$, then we have

$$\sup l_{n+1} \leq \frac{N_n + 1}{N_n} M_n \inf l_{n+1} = M_{n+1} \inf l_{n+1}$$

This gives (3.3) for all $n \geq 0$.

By (5.10) and (5.13), we obtain (5.8) with $C'' = e^{\xi}$.

We choose an arbitrary sequence $\{\eta_n\}_{n \in \mathbb{N}}$ that satisfy

$$(5.14) \qquad \sum_{n=0}^{\infty} \eta_n \; < \; \infty$$

$$(5.15) \qquad \text{and } \frac{\gamma}{2^n} < \eta_n \le \eta_0 \; < \; \gamma', \forall n \in \mathbb{N}$$

Let P be given by Lemma 3. We define for all $n \ge 0$

$$(5.16) \qquad \kappa_n \; = \; \bar{\lambda}_n^{-\eta_n}$$

$$(5.17) \qquad \chi_{n+1} \; = \; \inf_{\omega \in \Delta_n} \inf_{1 \le r \le r_n(\omega)} \left(\chi_n + \frac{r(\log \kappa_n - P \log C)}{\sum_{i=0}^{r-1} l_n(T_n^i(\omega))} \right)$$

$$(5.18) \qquad \bar{\lambda}_{n+1} \; = \; e^{\chi_{n+1} \inf l_{n+1}}$$

Now we are going to verify (5.6) and the second inequality in (5.7). We first show the following lemma.

LEMMA 6. *There exists $C' > 0$ such that we have for all sufficiently large $\bar{\lambda}_0 > 0$ the following*

$$(5.19) \qquad \chi_{n+1} \; > \; C' \chi_0$$
$$(5.20) \qquad \bar{\lambda}_n \; \ge \; \bar{\lambda}_0^{2^n}$$

for all $n \ge 0$.

As a consequence, for all sufficiently large $\bar{\lambda}_0$ we have (5.6) and the second inequality in (5.7) for all $n \ge 0$.

Proof. By (5.18) and Lemma 1 we have

$$(5.21) \qquad \log \bar{\lambda}_n \le \chi_n \inf l_n$$

for all $n \ge 0$.

Hence by (5.16), (5.17) and (5.21), we have for all $n \ge 0$,

$$(5.22) \qquad \begin{aligned} \chi_{n+1} \; &\ge \; \chi_n + \frac{1}{\inf l_n}(-\eta_n \log \bar{\lambda}_n - P \log C) \\ &\ge \; \chi_n + \frac{1}{\inf l_n}(-\eta_n \inf l_n \chi_n - P \log C) \\ &\ge \; \chi_n(1 - \eta_n) - P \log C \end{aligned}$$

Since by (5.9),(5.14) and (5.15), we have

$$\eta_n \; < \; \gamma' < 1, \forall n \ge 0$$

$$\text{and } \sum_{n=0}^{\infty} \eta_n \; < \; \infty$$

Then there exists $C' > 0, C'' > 0$ depending only on ξ, P, C, M_0, such that if $\chi_0 > C''$, then we have

(5.23) $$\chi_{n+1} > C'\chi_0, \forall n \geq 0$$

This proves (5.19) and (5.6).

To simplify the notations, by (5.22), we note that we can choose C'' to be large, still depending only on ξ, P, C, M_0 such that : if $\chi_0 > C''$, then

$$\chi_{n+1} \geq \chi_n(1 - e^{\xi}\gamma')$$

for all $n \geq 0$.

Then it is clear by (5.18) that for all $n \geq 0$

$$\bar{\lambda}_{n+1} = e^{\chi_{n+1} \inf l_{n+1}} \geq e^{\chi_n(1-e^{\xi}\gamma')N_n \inf l_n} \geq \bar{\lambda}_n^2$$

The last inequality follows from $N_n(1 - e^{\xi}\gamma') \geq 2$ by (5.9) and (5.11). This shows that we have

(5.24) $$\bar{\lambda}_n \geq \bar{\lambda}_0^{2^n}$$

This proves (5.20).

By (5.15), (5.16), (5.20) we have

$$\kappa_n = \bar{\lambda}_n^{-\eta_n} \leq \bar{\lambda}_0^{-2^n\eta_n} < \bar{\lambda}_0^{-\gamma}$$

This proves the second inequality in (5.7).

$\square$

Now we will define J_n inductively and verify (3.6) to (3.10) along the way.

For $n = 0$, we obtain (3.6) and (3.10) by Lemma 1 when λ is sufficiently large.

Assume that for $n \geq 0$, we have defined $J_0, \cdots, J_{n-1}$ and we have (3.6) and (3.10) for 0 to n. We define J_n by (5.1).

By (3.10) for n and (5.15), (5.16), for any $E \in I_{E_0} \setminus \bigcup_{n \geq m \geq 0} J_m$, for any $\tilde{\omega} \in \Delta_n$, we have

(5.25) $$\kappa_n > \bar{\lambda}_n^{-\gamma'}$$
(5.26) $$\lambda(B_n^E(\tilde{\omega})) \geq \bar{\lambda}_n$$

In particular, we see that the first inequality in (5.7) for n is valid.

We define that

(5.27) $$\zeta_n = C^P\bar{\lambda}_n^{-2+2\eta_n}N_n$$

By $c < 2 - 3\gamma'$, (5.16) and (5.20), for λ larger than some constant depending only on C, we have that

$$(5.28) \qquad \kappa_n^3 \bar{\lambda}_n^2 = \bar{\lambda}_n^{2-3\eta_n} \geq \bar{\lambda}_n^{2-3\gamma'} \geq \bar{\lambda}_0^{2^n c}$$

By (5.12), we see that

$$(5.29) \qquad N_n \leq 2^n N_0$$

Let $C_2 > 0$ be given by Lemma 3. Hence by (5.28) and (5.29), when $\bar{\lambda}_0$ is sufficiently large, we have

$$(5.30) \qquad \sup_{\omega \in \Delta_{n+1}} r_n(\omega) \leq N_n + 1 \leq C_2^{-1} \kappa_n^3 \bar{\lambda}_n^2$$

Combining (5.3), (5.4), (5.25), (5.26), (5.20) and (5.30), we see that the condition of Lemma 3 is satisfied for any $\omega \in \Delta_n$, any $1 \leq r \leq r_n(\omega)$ and any $E \in I_{E_0} \setminus \bigcup_{0 \leq m \leq n} J_m$ when $\bar{\lambda}_0$ is sufficiently large depending only on C.

Apply Lemma 3, we get (3.7), (3.8), (3.9) for n and (3.6), (3.10) for $n+1$ using (5.17) and (5.18). By induction, we see that (3.6) to (3.10) and the first inequality in (5.7) are valid for all $n \geq 0$.

Finally by (5.27), (5.29) we have that

$$\zeta_n \quad \leq \quad C^P \bar{\lambda}_n^{2\gamma'-2} 2^n N_0$$

By $0 < c < 2 - 3\gamma'$ and (5.20), when $\bar{\lambda}_0$ is sufficiently large depending only on C, we have

$$\zeta_n < \bar{\lambda}_n^{-c} \leq \bar{\lambda}_0^{-2^n c}$$

This proves (5.5).

By Lemma 1, we see that $\bar{\lambda}_0$ tends to infinity as λ tends to infinity. This concludes the proof. $\qquad \square$

6. COVER THE SPECTRUM

The goal of this section is to prove the following lemma, which shows that under suitable conditions the spectrum is covered by $\bar{J}$, where J is introduce in Notation 3.

LEMMA 7. *For any $0 < \gamma < \gamma' < \frac{1}{4}$, any $\gamma' < c < 2 - 3\gamma'$, for all sufficiently large λ, we define J_n and parameters $\bar{\lambda}_n, \zeta_n, \chi_n, M_n, N_n, \kappa_n$ that satisfy the conclusions of Lemma 5 with γ, γ', c. Then we have $\Sigma \bigcap I_{E_0} \subset \bar{J}$. Here J is defined by (5.2) in Section 5.*

Proof. By Lemma 5, for all sufficiently large λ, we have (3.3) to (3.10) and (5.5) to (5.8) for all $n \geq 0$.

Let $\hat{\chi} = \log \sup_{E \in I_{E_0}, \alpha \in \mathcal{A}} \|A_\alpha^E\|$. By (5.6), we see that there exists $C' > 0$ such that $\chi_n > C' \chi_0 > 0$ for all $n \geq 0$. Hence we can take $c_2 > 0$ be a constant so that

(6.1) $$\hat{\chi} < c_2 \chi_n, \forall n \geq 0$$

By the choice of χ_n in (3.9), we also see that

(6.2) $$\chi_n \leq \hat{\chi}, \forall n \geq 0$$

By ergodicity, for $\mu - a.e.\omega \in \Omega$, we can and do define

$$\begin{aligned} t_1(n, \omega) &= \inf\{k \geq 0; T^k(\omega) \in \Delta_n\} \\ t_2(n, \omega) &= \inf\{k > 0 : T^{-k}(\omega) \in \Delta_n\} \end{aligned}$$

such that $t_1(n, \omega), t_2(n, \omega) < \infty$. Then for such $\omega \in \Omega$, we define

$$\begin{aligned} W_1(n, \omega) &= T^{t_1(n,\omega)}(\omega) \\ W_2(n, \omega) &= T^{-t_2(n,\omega)}(\omega) \end{aligned}$$

It is direct to see that

(6.3) $$W_1(n, \omega) = T_n(W_2(n, \omega)) \text{ for } \mu - a.e.\omega \in \Omega$$
(6.4) $$t_1(n, \omega) \leq \sup l_n$$

Since (Ω, T, μ) is ergodic, it is a standard fact that (Δ_n, T_n, μ_n) is also ergodic. Hence for $\mu_n - a.e.\omega \in \Delta_n$, we can and do define

$$\begin{aligned} s_1(n, \omega) &= \inf\{k \geq 0; T_n^k(\omega) \in \Delta_{n+1}\} \\ s_2(n, \omega) &= \inf\{k \geq 0; T_n^{-k}(\omega) \in \Delta_{n+1}\} \end{aligned}$$

such that $s_1(n, \omega), s_2(n, \omega) < \infty$. Then for such $\omega \in \Omega$, we define

$$\begin{aligned} W_3(n, \omega) &= T_n^{s_1(n,\omega)}(\omega) \\ W_4(n, \omega) &= T_n^{-s_2(n,\omega)}(\omega) \end{aligned}$$

We define

$$\Omega_n = \{\omega \in \Omega; s_2(n, \omega_2(n, \omega)), s_1(n, \omega_1(n, \omega)) > 2N_n^{\frac{1}{2}}\}$$

By the definition of $s_1(\cdot, \cdot), s_2(\cdot, \cdot)$, for any $\omega \in \Delta_n$ such that $s_1(n, \omega) > 0$ or $s_2(n, \omega) > 0$, we have $\omega \in \Delta_n \setminus \Delta_{n+1}$. Hence for any $\omega \in \Omega_n$, we have that $\omega_1(n, \omega), \omega_2(n, \omega) \in \Delta_n \setminus \Delta_{n+1}$.

LEMMA 8. *There exists a constant $c_4 > 0$ such that $\mu(\Omega_n) > c_4$ for all sufficiently large n.*

Proof. By (3.4), for all $m \geq 0$, $\Delta_{m+1}, T_m(\Delta_{m+1}), \cdots, T_m(^{N_m-1}\Delta_{m+1})$ are mutually disjoint and belong to Δ_m. Moreover it is easy to see that their union takes up a proportion of Δ_m no less than $\frac{N_n}{N_n+1}$. We see from (P2) that $T_0^{k_0} T_1^{k_1} \cdots T_{n-1}^{k_{n-1}} T_n^{k_n} \Delta_{n+1}$ for $0 \leq k_0 \leq N_0 - 1, 0 \leq k_1 \leq N_1 - 1, \cdots, 0 \leq k_{n-1} \leq N_{n-1}, 2N_n^{\frac{1}{2}} \leq k_n \leq N_n - 2N_n^{\frac{1}{2}}$ all belong to $\Omega_n \bigcap \Delta_0$, and are mutually disjoint for points in different sets have different landing time with respect to sequence $\Delta_1, \cdots, \Delta_{n+1}$. We know that $\mu(\Omega_n) \geq \mu(\Delta_0)\mu_0(\Omega_n \bigcap \Delta_0) > \mu(\Delta_0)\frac{N_n - 4N_n^{\frac{1}{2}}}{N_n+1} \frac{N_{n-1}}{N_{n-1}+1} \cdots \frac{N_0}{N_0+1}$. This proves the lemma since we have chosen N_n so that $\prod_{n=0}^{\infty} \frac{N_n}{N_n+1} > 0$. $\qquad\square$

Define

$$\Omega' = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \Omega_m$$

By Lemma 8, we have

$$\mu(\Omega') > 0$$

In order to prove Lemma 7, it suffices to prove that :
For all $\omega \in \Omega'$, we have

$$\Sigma_\omega \bigcap I_{E_0} \subset \overline{J}$$

Assume the contrary, then there exists $\omega \in \Omega'$ such that

$$\nu_\omega(I_{E_0} \setminus \overline{J}) > 0$$

Here ν_ω is the spectral measure of the Schrödinger operator associated to ω.

Then by Theorem 7, there exists $E \in I_{E_0} \setminus \overline{J}$ such that the Schrödinger operator with potential $\{v(T^n(\omega)_0)\}_{n \in \mathbb{Z}}$ admitting a generalized eigenfunction with polynomial growth (in fact, of degree 1 since we are considering a one-dimensional operator, but this point is not essentially used as we can see from the proof). Thus there exists $h \in \mathbb{R}^2, c_3 > 0$, such that

$$(6.5) \qquad \|A^E(m,\omega)h\| \leq c_3(1+|m|)\|h\|, \forall m \in \mathbb{Z}$$

By the definition of Ω', there exists arbitrarily large n, such that $\omega \in \Omega_n$. Denote

$$
\begin{aligned}
\omega^1 &= W_1(n,\omega), & \omega^2 &= W_2(n,\omega) \\
\omega^3 &= W_3(n,\omega^1), & \omega^4 &= W_4(n,\omega^2)
\end{aligned}
$$

and

$$t_1 \;=\; t_1(n,\omega), \quad s_1 = s_1(n,\omega^1), \quad s_2 = s_2(n,\omega^2)$$

We verify by definition that

(6.6)
$$\omega^3 \;=\; T_n^{s_1}(\omega^1)$$

and

$$T_n(\omega^2) = \omega^1$$

We also denote

$$g \;=\; A^E(t_1,\omega)h$$

Denote the argument of g by $\theta(g)$. More precisely we have

$$g = \|g\| \begin{bmatrix} \cos\theta(g) \\ \sin\theta(g) \end{bmatrix}$$

By (6.4) we have estimate

(6.7)
$$\begin{aligned}
\|g\| &\geq\; \|A^E(t_1,\omega)\|^{-1}\|h\| \\
&\geq\; e^{-t_1\hat{\chi}}\|h\| \\
&\geq\; e^{-\sup l_n\hat{\chi}}\|h\|
\end{aligned}$$

Since $\omega \in \Omega_n$, we have $s_1, s_2 > 0$. By definition

$$\omega^2 = T_n^{s_2}(\omega^4)$$

There exist $1 \leq i_1, i_2 \leq K$, such that

$$\omega^1 \in \Delta_{(i_1)}, \qquad \omega^4 \in \Delta_{(i_2)}$$

We denote that

(6.8)
$$L_1 \;=\; \sum_{i=0}^{s_1-1} l_n(T_n^i(\omega^1)) - i_1$$

(6.9)
$$L_2 \;=\; \sum_{i=0}^{s_2} l_n(T_n^i(\omega^4)) - i_2$$

By (P5), we have

$$\begin{aligned}
A^E(L_1, T^{i_1}(\omega^1)) &=\; A_n^E(T_n^{s_1-1}(\omega^1)) \cdots A_n^E(T_n(\omega^1))B_n^E(\omega^1) \\
A^E(-L_2, \omega^1)^{-1} &=\; A_n^E(T_n^{s_2}(\omega^4)) \cdots A_n^E(T_n(\omega^4))B_n^E(\omega^4)
\end{aligned}$$

Denote

$$\begin{aligned}
G_1^E &=\; A^E(L_1, T^{i_1}(\omega^1)) \\
G_2^E &=\; A^E(-L_2, \omega^1)^{-1} \\
C(E) &=\; C_n^E(\omega^1) = A^E(i_1, \omega^1)
\end{aligned}$$

By (3.6) we know that $G_1^E, G_2^E \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$, and by (3.9) and (6.8),(6.9) we have

$$(6.10) \qquad\qquad \|G_1^E\| \geq e^{(L_1+i_1)\chi_{n+1}}$$

$$(6.11) \qquad\qquad \|G_2^E\| \geq e^{(L_2+i_2)\chi_{n+1}}$$

Denote

$$\begin{aligned} u^1 &= u(G_1^E), & s^1 &= s(G_1^E) \\ u^2 &= u(G_2^E), & s^2 &= s(G_2^E) \end{aligned}$$

Then by (3.7),(3.8) and (5.5), we have

$$|s^1 - s(B_n^E(\omega^1))|_{\mathbb{R}/\pi\mathbb{Z}}, |u^2 - u(B_n^E(\omega^2))|_{\mathbb{R}/\pi\mathbb{Z}} \leq \zeta_n < \bar{\lambda}_n^{-c}$$

Since $E \notin J$, by (5.4) we have either

$$(6.12) \qquad\qquad B_n^E(\omega^1), B_n^E(\omega^2) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$$

$$(6.13) \quad \text{and} \angle(R_{\frac{\pi}{2}-s(B_n^E(\omega^1))}C(E)R_{u(B_n^E(\omega^2))} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \geq \kappa_n$$

or

$$B_n^E(\omega^1) \in SO(2, \mathbb{R}) \text{ or } B_n^E(\omega^2) \in SO(2, \mathbb{R})$$

The second alternate contradicts (3.6). Indeed, we can apply (3.6) to E, ω_4, $q = s_2 - 1$ and $r = s_2$; then again apply to $q = s_2$ and $r = s_2 + 1$. Thus we have (6.12) and (6.13).

By (5.7), we have $\kappa_n \geq \bar{\lambda}_n^{-\gamma'}$. By $c > \gamma'$ and Lemma 2 applied to $u^2, s^1, u(B_n^E(\omega^2)), s(B_n^E(\omega^1))$, when $\bar{\lambda}_0$ is bigger than some absolute constant, we can ensure that

$$(6.14) \quad \angle(R_{\frac{\pi}{2}-s^1}C(E)R_{u^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) > \kappa_n - c_5\bar{\lambda}_n^{-c} \geq \frac{1}{2}\kappa_n$$

We distinguish two cases:
(1)If we have $|\theta(g) - u^2|_{\mathbb{R}/\pi\mathbb{Z}} < \frac{1}{10}c_5^{-1}\kappa_n$.
Then by (6.14) and Lemma 2, we have

$$\angle(R_{\frac{\pi}{2}-s^1}C(E) \begin{bmatrix} \cos\theta(g) \\ \sin\theta(g) \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \angle(R_{\frac{\pi}{2}-s^1}C(E)R_{\theta(g)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) > \frac{1}{4}\kappa_n$$

In this case, when n is larger than some constant depending only on c_2, we have

$$
\begin{aligned}
\|A^E(L_1 + i_1, \omega^1)g\| &= \|A^E(L_1, T^{i_1}(\omega^1))A^E(i_1, \omega^1)g\| \\
&= \|G_1^E C(E)g\| \\
&= \left\| R_{u^1} \begin{bmatrix} \|G_1^E\| & 0 \\ 0 & \|G_1^E\|^{-1} \end{bmatrix} R_{\frac{\pi}{2}-s^1} C(E)g \right\| \\
&\gtrsim c_5^{-1}C^{-1}e^{(L_1+t_1)\chi_{n+1}}\kappa_n \|g\| \text{ (by (6.10))} \\
&\geq c_5^{-1}C^{-1}e^{(L_1+t_1)\chi_{n+1}-\gamma'\chi_n \inf l_n} \|g\| \text{ (by (5.7))}
\end{aligned}
$$

(6.15)

Since $\omega \in \Omega_n$, we have

$$
(6.16) \qquad s_1 \geq 2N_n^{\frac{1}{2}}
$$

By (6.8) it is clear that

$$
(6.17) \qquad s_1 \sup l_n \geq L_1 \geq s_1 \inf l_n - i_1
$$

Then by (6.17), (6.16), for all large n we have

$$
(6.18) \qquad L_1 \geq \frac{2}{3}s_1 \inf l_n
$$

Moreover by (6.1) and (6.2), we have

$$
\chi_{n+1} \geq c_2^{-1}\hat{\chi} \geq c_2^{-1}\chi_n
$$

Hence by (6.15) and (6.16) we have for all sufficiently large n that

$$
\|A^E(L_1 + i_1, \omega^1)g\| \gtrsim c_5^{-1}C^{-1}e^{\frac{1}{2}L_1\chi_{n+1}} \|g\|
$$

By (6.16) and (5.8), for n sufficiently large we have

$$
(6.19) \qquad s_1 > 12C''M_0c_2 \geq 12M_nc_2
$$

where C'' is given by Lemma 5.

Thus

$$
\begin{aligned}
\|A^E(L_1 + i_1 + t_1, \omega)h\| &= \|A^E(L_1 + i_1, \omega^1)g\| \\
&\gtrsim c_5^{-1}C^{-O(1)}e^{\frac{1}{2}L_1\chi_{n+1}-\sup l_n\hat{\chi}} \|h\|
\end{aligned}
$$

By (6.18),(6.1) and (6.19), we have

$$
\begin{aligned}
\frac{1}{2}L_1\chi_{n+1} - \sup l_n\hat{\chi} &\geq (\frac{1}{3}s_1 \inf l_n - c_2 \sup l_n)\chi_{n+1} \\
&\geq \frac{1}{4}s_1 \inf l_n\chi_{n+1}
\end{aligned}
$$

By $1 \leq i_1 \leq K$, (6.4),(6.17), (3.3), (5.6) and (5.8)

$$s_1 \inf l_n \chi_{n+1} \geq \frac{1}{M_n} s_1 \sup l_n \chi_{n+1}$$

$$\geq \frac{1}{O(M_n)}(L_1 + i_1 + t_1)\chi_0 \geq \frac{1}{O(M_0)}(L_1 + i_1 + t_1)\chi_0$$

Thus we have

$$\|A^E(L_1 + i_1 + t_1, \omega)h\| \gtrsim c_5^{-1} C^{-1} e^{\frac{1}{O(M_0)}(L_1+i_1+t_1)\chi_0}\|h\|$$

This contradicts (6.5) when n is large.

(2)If we have $|\theta(g) - u^2|_{\mathbb{R}/\pi\mathbb{Z}} \geq \frac{1}{10}c_5^{-1}\kappa_n$

Since

(6.20) $u^2 = s(A^E(-L_2, \omega^1))$

Similar computations shows that for all sufficiently large $n \geq 0$ we have

$$\|A^E(-L_2, \omega^1)g\| = \|(G_2^E)^{-1}g\|$$
$$\gtrsim c_5^{-1} C^{-1} e^{L_2 \chi_{n+1}} \kappa_n \|g\|$$
$$\gtrsim c_5^{-1} C^{-1} e^{\frac{1}{2}L_2 \chi_{n+1}} \|g\|$$

and we can reach a contradiction in a way similar to (1). This proves the statement in the lemma.

□

7. AREA OF THE SPECTRUM AND THE PROOF OF THEOREM 3

To prove Theorem 8, and as a consequence, Theorem 3, it remains to estimate the measure of $\bar{J}$, where J is defined in (5.2) in Section 5.

NOTATION 4. *For any $n \geq 0$, any $\alpha \in \mathcal{A}_n$ such that $\alpha = \omega_i \omega_{i+1} \cdots \omega_{l_n(\omega)-1}$ for some $1 \leq i \leq K$ and $\omega \in \Delta_{(i)}$, for each $0 \leq m \leq n-1$, we define*

$$hd_m(\alpha) = \omega_i \omega_{i+1} \cdots \omega_{l_m(\omega)-1}$$
$$rr_m(\alpha) = \tilde{\omega}_j \tilde{\omega}_{j+1} \cdots \tilde{\omega}_{l_m(\tilde{\omega})-1}$$

Here $\tilde{\omega} = T_m^{-1} T_n(\omega)$ such that $\tilde{\omega} \in \Delta_m \bigcap \Delta_{(j)}$ for some $1 \leq j \leq K$. We can verify by (P2) that $hd_m(\alpha), rr_m(\alpha)$ are respectively prefix and suffix of α, and belong to $\mathcal{A}_m$.

The following estimate is essentially proved in [1] (see also [11]) by explicit calculations. Here we give a sketched proof.

LEMMA 9. *There exists $C_1 > 0$ such that for all sufficiently large λ the following is true. For all $\alpha, \beta \in \mathcal{A}_0$, any $1 \le j \le K$, any $E \in I_{E_0}$ we have $A_\alpha^E, A_\beta^E \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$. As functions from I_{E_0} to $\mathbb{R}/\pi\mathbb{Z}$, $E \mapsto s(A_\beta^E)$ and $E \mapsto u(A_\alpha^E)$ are C^1. Moreover, consider $E \mapsto R_{\frac{\pi}{2}-s(A_\beta^E)} C_j^E R_{u(A_\alpha^E)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a function from I_{E_0} to $\mathbb{P}\mathbb{R}^2$, we have*

$$\left| \partial_E \left(R_{\frac{\pi}{2}-s(A_\beta^E)} C_j^E R_{u(A_\alpha^E)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right| > C_1$$

Proof. Since for any $\alpha \in \mathcal{A} \setminus \{\alpha_0\}$, any $E \in I_{E_0}$, we have $|tr(A_\alpha^E)| \ge \lambda - H$. When $\lambda > H + 2$, we have $A_\alpha^E \in SL(2, \mathbb{R}) \setminus SO(2, R)$.

Denote

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_\beta(E) = s(A_\beta^E), u_\alpha(E) = u(A_\alpha^E)$$

It well-known that under the condition of the lemma, s_β, u_α are C^1.

We have

(7.1) $\partial_E (R_{\frac{\pi}{2}-s_\beta(E)} C_j^E R_{u_\alpha(E)} v)$

$= \partial_E R_{\frac{\pi}{2}-s_\beta(E)} (C_j^E R_{u_\alpha(E)} v) + DR_{\frac{\pi}{2}-s_\beta(E)} (C_j^E R_{u_\alpha(E)} v) \partial_E C_j^E (R_{u_\alpha(E)} v)$

$+ D(R_{\frac{\pi}{2}-s_\beta(E)} C_j^E)(R_{u_\alpha(E)} v) \partial_E R_{u_\alpha(E)}(v)$

Here and the following, the derivatives of varies functions from E to $\mathbb{P}\mathbb{R}^2$ are interpreted through identifying $\mathbb{R}/\pi\mathbb{Z}$ with $\mathbb{P}\mathbb{R}^2$ as

$$\theta \in \mathbb{R}/\pi\mathbb{Z} \mapsto \mathbb{R} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \in \mathbb{P}\mathbb{R}^2$$

Since $SL(2, \mathbb{R})$ act $\mathbb{P}R^2$ through smooth, orientation preserving diffeomorphisms, for any $M \in SL(2, \mathbb{R})$, any $\psi \in \mathbb{P}\mathbb{R}^2$, we have

$$DM(\psi) > 0$$

It is shown in [1] (see also [11]) that

$$\partial_E R_{\frac{\pi}{2}-s_\beta(E)}(\phi) \le 0$$
$$\partial_E R_{u_\alpha(E)}(\phi) \le 0$$

for all $\phi \in \mathbb{P}\mathbb{R}^2$. So the first term and the last term in (7.1) is non-positive.

Since for any $\phi \in \mathbb{P}\mathbb{R}^2$, we have

$$DR_{\frac{\pi}{2}-s_\beta(E)}(\phi) = 1$$

it remains to check that $\partial_E C_j^E(R_{u_\alpha(E)}v)$ is uniformly bounded by a negative constant for all choice of $\alpha \in \mathcal{A}, 1 \leq j \leq K$ and $E \in I_{E_0}$.

Denote $\phi = R_{u_\alpha(E)}v$. It is well-known that for any $\psi \in \mathbb{PR}^2$, we have

$$\partial_E C^E(\psi) \leq 0$$

Then we have

$$\partial_E C_j^E(\phi) = \sum_{i=0}^{j-1} DC_i^E(C_{j-i}^E(\phi))\partial_E C^E(C_{j-i-1}^E(\phi)) \leq DC_{j-1}^E(C^E(\phi))\partial_E C^E(\phi)$$

When λ is sufficiently large, $u_\alpha(E)$ can be made arbitrarily close to 0 for all $E \in I_{E_0}$. Then we can ensure that ϕ is close to v so that

$$\partial_E C_k^E(\phi) \leq \frac{1}{2}DC_{k-1}^E(C^E(\phi))\partial_E C^E(v)$$

Straight-forward computation shows that the right hand is strictly negative. This completes the proof.

□

Now we are going to show that the Lebesgue measure of $\overline{J}$ is small.

LEMMA 10. *For any $0 < \gamma < \gamma' < \frac{1}{4}$, any $\gamma' < c < 2 - 3\gamma'$, there exists $C_3 > 0$, such that the following is true. For λ sufficiently large, we define J_n and parameters $\bar{\lambda}_n, \zeta_n, \chi_n, M_n, N_n, \kappa_n$ that satisfy the conclusions in Lemma 5 with γ, γ', c. Define J by (5.1). Then $Leb(\overline{J}) \leq C_3\lambda^{-\gamma}$*

Proof. By Lemma 5, for all sufficiently large λ, we have (3.3) to (3.10) and (5.5) to (5.8) for all $n \geq 0$.

For any $n \geq 1$, for any $\alpha \in \mathcal{A}_n$, for any $E \in I_{E_0}$, if $\alpha = \omega_i \omega_{i+1} \cdots \omega_{l_n(\omega)-1}$ for some $\omega \in \Delta_n \cap \Delta_{(i)}$ and $1 \leq i \leq K$, by definition we have

$$\begin{aligned} A^E(\alpha) &= B_n^E(\omega) \\ A^E(hd_{n-1}(\alpha)) &= B_{n-1}^E(\omega) \\ A^E(rr_{n-1}(\alpha)) &= B_{n-1}^E(T_{n-1}^{r_{n-1}(\omega)-1}(\omega)) \end{aligned}$$

Then for any $\alpha, \beta \in \mathcal{A}_n$, for any $1 \leq i \leq K$, for any $E \in J(\alpha, \beta, i, \kappa_n) \setminus \bigcup_{0 \leq j \leq n-1} J_j$, by (3.6) we have

$$A^E(hd_j(\beta)), A^E(rr_j(\alpha)) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R}), \forall 0 \leq j \leq n-1$$

For any $n \geq 1$, any $\alpha, \beta \in \mathcal{A}_n$, any $E \in I_{E_0} \setminus \bigcup_{0 \leq m \leq n-1} J_m$, by (3.7) we know that

$$(7.2) \quad |u(A^E(\alpha)) - u(A^E(rr_{n-1}(\alpha)))|_{\mathbb{R}/\pi\mathbb{Z}} < \zeta_{n-1} < \bar{\lambda}_0^{-2^{n-1}c}$$

by (3.8)

$$(7.3) \qquad |s(A^E(\beta)) - s(A^E(hd_{n-1}(\beta)))|_{\mathbb{R}/\pi\mathbb{Z}} \ < \ \bar{\lambda}_0^{-2^{n-1}c}$$

Moreover by Lemma 2, (7.2), (7.3) for all $1 \leq m \leq n$, we see that for any $E \in J(\alpha, \beta, i, \kappa_n) \setminus \bigcup_{0 \leq j \leq n-1} J_j$ we have the following

$$
\begin{aligned}
\bar{\lambda}_0^{-\gamma} \ \geq \ &\kappa_n \geq \angle(R_{\frac{\pi}{2} - s(A^E(\beta))} C_i^E R_{u(A^E(\alpha))} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\
\geq \ &\angle(R_{\frac{\pi}{2} - s(A^E(hd_{n-1}(\beta)))} C_i^E R_{u(A^E(rr_{n-1}(\alpha)))} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) - c_5 \bar{\lambda}_0^{-2^{n-1}c} \\
&\cdots \\
\geq \ &\angle(R_{\frac{\pi}{2} - s(A^E_{hd_0(\beta)})} C_i^E R_{u(A^E_{rr_0(\alpha)})} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) - c_5(\bar{\lambda}_0^{-c} + \cdots + \bar{\lambda}_0^{-2^{n-1}c})
\end{aligned}
$$

Then

$$\angle(R_{\frac{\pi}{2} - s(A^E_{hd_0(\beta)})} C_i^E R_{u(A^E_{rr_0(\alpha)})} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \leq c_5(\bar{\lambda}_0^{-c} + \cdots + \bar{\lambda}_0^{-2^{n-1}c}) + \bar{\lambda}_0^{-\gamma}$$

Denote $\theta = \sup_{n \geq 0}(c_5(\bar{\lambda}_0^{-c} + \cdots + \bar{\lambda}_0^{-2^{n-1}c}) + \bar{\lambda}_0^{-\gamma})$. Then we have that $E \in J(rr_0(\alpha), hd_0(\beta), j, \theta)$.

Then for all $n \geq 0$, $\alpha, \beta \in \Delta_n$, any $1 \leq i \leq K$ we have the following

$$(J(\alpha, \beta, i, \kappa_n) \setminus \bigcup_{0 \leq j \leq n-1} J_j) \subset \overline{J(rr_0(\alpha), hd_0(\beta), i, \theta)}$$

Take the unions of the above expression for all $n \geq 0$, $\alpha, \beta \in \Delta_n$, all $1 \leq i \leq K$, we obtain

$$J \subset \bigcup_{\alpha, \beta \in \mathcal{A}_0, 1 \leq j \leq K} \overline{J(\alpha, \beta, j, \theta)}$$

The right hand side is a closed set and by Lemma 9 and (3.11), it is of measure $O(\theta)$. Since $c > \gamma$, then there exists a constant $Q > 0$ depending only on γ, γ', c such that $\theta < Q\lambda^{-\gamma}$ for all λ sufficiently large. This concludes the proof. $\square$

Proof of Theorem 8. For any $\gamma \in (0, \frac{1}{4})$, we can choose $\gamma' \in (\gamma, \frac{1}{4})$ and $c \in (\gamma', 2 - 3\gamma')$. Then when λ is sufficiently large, Theorem 8 follows from Lemma 7 and Lemma 10. When λ is small, we use the trivial bound $Leb(\Sigma_v \bigcap I_{E_0}) \leq Leb(I_{E_0}) \leq 2H$. After possibly enlarging Q, we obtain Theorem 8. $\square$

8. Proof of Theorem 4 and Theorem 5

8.1. **Proof of Theorem 4.** The construction of the required subshift follows closely the proof of Theorem 1 in [1]. We refer to [1] for some relevant lemmata. Without loss of generality, let us assume that B is a countably infinite set of potentials and $0 < \epsilon < 1$. We will inductively define collections of finite words S_n, subshifts Ω_n, closed subsets $\Sigma_{n,m}$ for $1 \le n \le m$.

For $n = 1$, we define

$$(8.1) \qquad\qquad S_1 = \{1, \cdots, k\}$$

We define Ω_1 to be the two-sided infinite concatenations of the words in S_1. We now pick any element $v_1 \in B$. For each word $w \in S_1$, we denote the spectrum of the periodic potential associated to v_1 and w by $\Sigma_{1,1}(w)$, and define

$$(8.2) \qquad\qquad \Sigma_{1,1} = \bigcup_{w \in S_1} \Sigma_{1,1}(w)$$

Assume $S_n, \Omega_n, \Sigma_{i,n}, \forall 1 \le i \le n$ are constructed. We denote

$$S_n = \{w_{n,1}, w_{n,2}, \cdots, w_{n,k_n}\}$$

For any given integer $N_n \ge 1$, we define

$$S_{n+1} = \{w_{n,1}w_{n,2}\cdots w_{n,k_n}w_{n,k}^l; 1 \le k \le k_n, N_n \le l < N_n + N_n^{\frac{1}{2}\epsilon}\}$$

and define Ω_{n+1} to be the two-sided infinite concatenation of the words in S_{n+1}. It is direct to see that $\Omega_{n+1} \subset \Omega_n$.

We pick any element $v_{n+1} \in B \setminus \{v_1, \cdots, v_n\}$. For each $1 \le i \le n+1$, for each $w \in S_{n+1}$, we denote the spectrum of the periodic potential associated to v_i and w by $\Sigma_{i,n+1}(w)$, and denote

$$(8.3) \qquad\qquad \Sigma_{i,n+1} = \bigcup_{w \in S_{n+1}} \Sigma_{i,n+1}(w)$$

It is clear that $Leb(\Sigma_{n+1}) > 0$. By a slightly modified version of Lemma 1 in [1], we can choose a positive integer N_n depending only on $S_n, \Omega_n, \Sigma_{i,n}$ such that the following is true.

$$(8.4) \qquad\qquad Leb(\Sigma_{i,n} \setminus \Sigma_{i,n+1}) < Leb(\Sigma_{i,i})2^{-(n+1)}$$

for any $1 \le i \le n$. We define $\Omega = \bigcap_n \Omega_n$. For each $v \in B$, denote the spectrum associated to Ω and v by Σ. For some $i \in \mathbb{N}$, we have $v = v_i$. Then following [1], we have

$$(8.5) \qquad\qquad \Sigma \supseteq \limsup_{n \to \infty} \Sigma_{i,n}$$

Then by the same reasoning in [1], we have $Leb(\Sigma) > \frac{1}{2}Leb(\Sigma_{i,i}) > 0$. Following the proof of Lemma 2 in [1], we can show that Ω is minimal and aperiodic.

It remains to show that when N_n are properly chosen, we can ensure that Ω has required complexity function.

For any $n \geq 0$, define

$$(8.6) \qquad M_n = \min\{|w|; w \in S_n\}, P_n = \max\{|w|; w \in S_n\}$$

It is direct to see that

$$(8.7) \qquad M_{n+1} \geq N_n M_n$$

$$(8.8) \qquad P_{n+1} \leq (N_n + N_n^{\frac{1}{2}\epsilon})P_n$$

$$(8.9) \qquad S_{n+1} = N_n^{\frac{1}{2}\epsilon}|S_n|$$

Hence for any $n \geq 0$

$$(8.10) \qquad S_n \lesssim M_n^{\frac{1}{2}\epsilon}$$

$$(8.11) \qquad \frac{P_{n+1}}{M_{n+1}} \leq (1 + N_n^{-1+\frac{1}{2}\epsilon})\frac{P_n}{M_n}$$

From the construction, we see that we can also ensure that

$$(8.12) \qquad \sum_{n \geq 0}^{\infty} N_n^{-1+\frac{1}{2}\epsilon} < \infty$$

Then there exists $C > 0$ such that for any $n \geq 0$, we have

$$(8.13) \qquad P_n \leq CM_n$$

For any $L \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $M_n \leq L < M_{n+1}$. For any word w of length L, there exists two words $w_1, w_2 \in S_{n+1}$, such that w is a subword of the concatenation $w_1 w_2$ and is not a subword of w_1. Assume

$$(8.14) \qquad w_1 = w_{n,1} \cdots w_{n,k_n} w_{n,i}^l$$

$$(8.15) \qquad w_2 = w_{n,1} \cdots w_{n,k_n} w_{n,j}^m$$

We have four possibilities:

(1) w does not intersect $w_{n,1} \cdots w_{n,k_n}$. Then w is a subword of $w_{n,j}^m$. Then there are at most $|S_n|L$ possible choices of w;

(2) w contains $w_{n,1} \cdots w_{n,k_n}$. Then w is the concatenation of a suffix of $w_{n,i}^l$ (possibly empty), $w_{n,1} \cdots w_{n,k_n}$ and a prefix of $w_{n,j}^m$. In this case, there are at most $|S_n|^2 L$ possible choices of w;

(3) w intersect both $w_{n,1} \cdots w_{n,k_n}$ and $w_{n,j}^m$. Then w is determined by a prefix of $w_{n,j}^m$ of length at most L. There are at most $|S_n|L$ possible choices of w;

(4) w is contained in $w_{n,1} \cdots w_{n,k_n}$. Then there are at most $P_n|S_n|$ possibilities. Since $P_n \leq CM_n \leq CL$, we have at most $CL|S_n|$ possibilities.

Combining all three cases, we have

$$(8.16) \quad p(L) \leq |S_n|L + |S_n|^2L + |S_n|L + C|S_n|L \lesssim L^{1+\epsilon}$$

This proves the theorem.

8.2. **Proof of Theorem 5.** Fix $E \in (-2 + v(i), 2 + v(i))$, then A_i^E is an elliptic matrix. Assume to the contrary that $E \notin \Sigma_v$. Then we can take an open interval neighbourhood of E, denoted by J, such that $J \subset (-2 + v(i), 2 + v(i)) \bigcap \Sigma_v^c$. By Theorem 6 the cocycle A^E over Ω is Uniformly Hyperbolic. Thus we can define stable, unstable directions, denoted respectively by $s, u : \Omega \to \mathbb{PR}^2$. After possibly reducing J, we can assume that for any $E' \in J$, we have $s(E'), u(E') : \Omega \to \mathbb{PR}^2$, and for any $\omega \in \Omega$, the function $s(\cdot, \omega), u(\cdot, \omega) : J \to \mathbb{PR}^2$ are C^1 (in fact analytic) and the C^1 norm of these functions are bounded uniform in $\omega \in \Omega$. We take any $\omega \in \Omega$ such that $\omega_0 = \cdots = \omega_{N-1} = i$, where N will be chosen to be large. Denote $\omega' = T^N(\omega)$. Then $s(E', \omega') = (A_i^{E'})^N s(E', \omega)$ for all $E' \in J$. Straightforward calculation shows that the C^1 norm of $s(\cdot, \omega')$ will be $\Theta(N)$. When N is large, we have a contradiction. Hence $E \in \Sigma_v$. This proves the theorem.

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